A[nother] beamforming strategy

an information theoretic look at beamforming with PAFs

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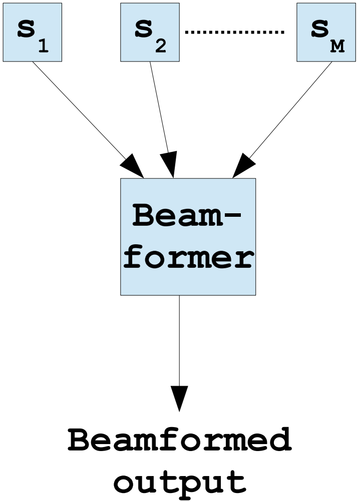
What is beamforming?

- *Beam* forming = data reduction
- Exploit sensors' covariance
- Sensors can be of various nature:
  - Non-co-located antennas
  - Time samples
  - Pixels
  - Frequency channels
  - ...
- Beamformers can be of various nature:
  - Spatial beamformers
  - Time-domain digital filters
  - Spatial (image) filters
  - Cyclic filters
  - ...



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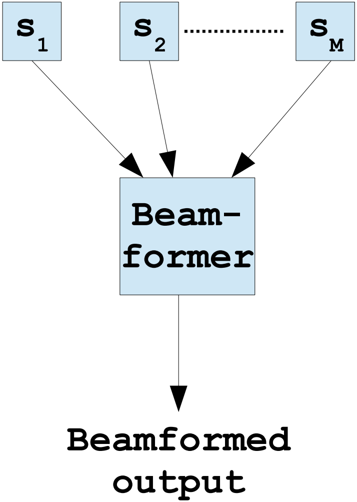

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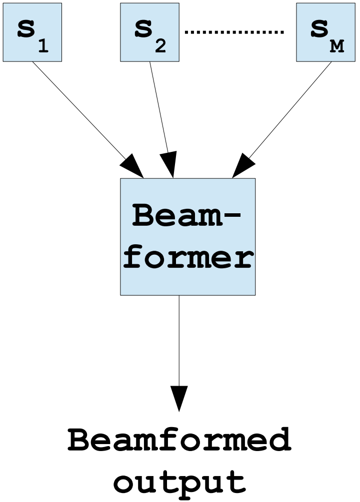

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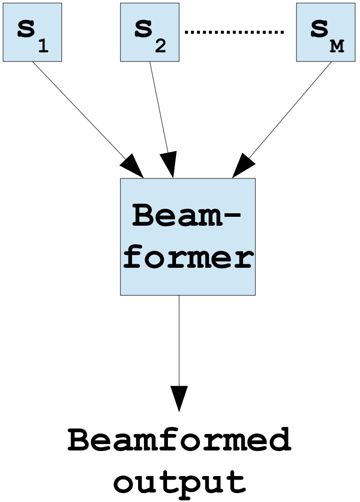
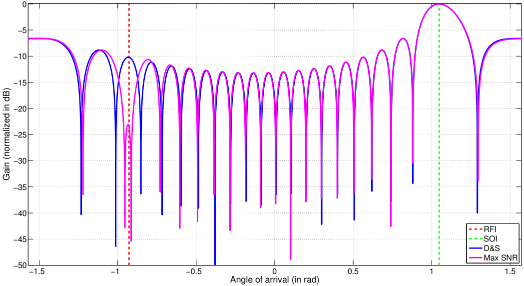


Adaptive beamforming

- Deterministic methods
- Statistical methods

$$\begin{aligned} \mathbf{w}_{\text{adapt}} = & \underset{\mathbf{w}}{\text{argmax}} \quad \phi\left[\mathbf{w}, \mathbf{x}(t), \ldots\right] \\ & \text{subject to} \quad \psi\left[\mathbf{w}, \mathbf{x}(t), \ldots\right] \end{aligned}$$

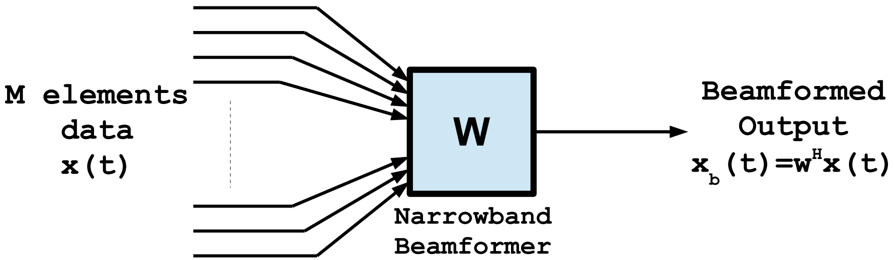


Examples:

- Max SNR
- MSC
- LCMV

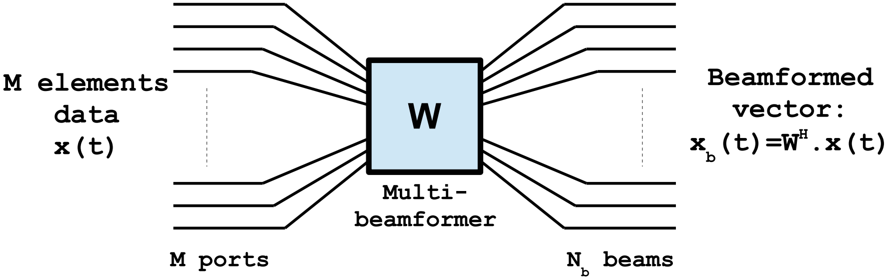
Adaptive beamforming

Mono-beam system:

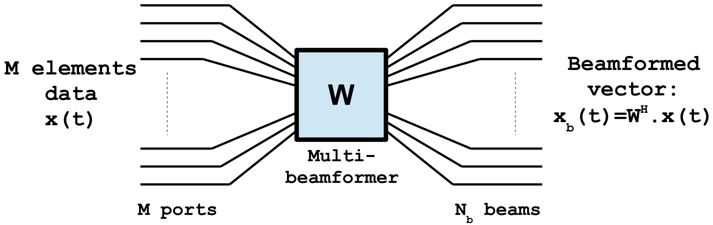


Adaptive beamforming

Multi-beam system:



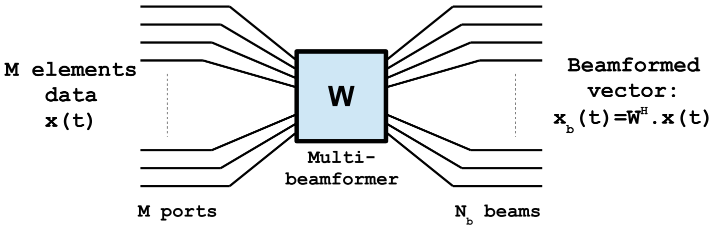
Multi-beamforming approach



$$\mathbf{W} = \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \cdots & \mathbf{w}_{N_b} \end{bmatrix}$$ beamforming *matrix*

with : $$\mathbf{w}_k = \begin{bmatrix} w_{k_1} & w_{k_2} & \cdots & w_{k_M} \end{bmatrix}^T$$ beamforming *vector*

Multi-beamforming approach



$$\mathbf{W}_{\text{opt}} = \underset{\mathbf{W}}{\text{argmax}}\ \text{f}(?, \mathbf{W})$$

e.g.

$$\mathbf{W}_{\text{opt}} = \underset{\mathbf{W}}{\text{argmax}}\ \mathbf{SNR}\left(\mathbf{x}(t), \mathbf{W}, \begin{bmatrix}\theta_{\text{noise}}\\ \phi_{\text{noise}}\end{bmatrix}, \begin{bmatrix}\theta_1\\ \phi_1\end{bmatrix}, \ldots, \begin{bmatrix}\theta_{N_b}\\ \phi_{N_b}\end{bmatrix}\right)$$

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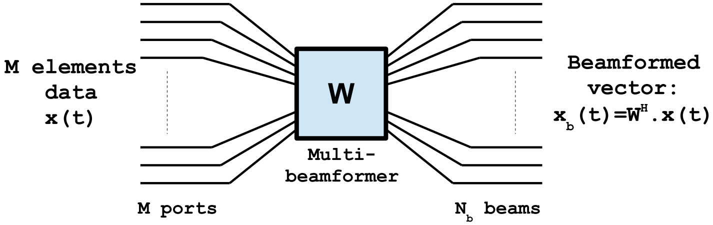

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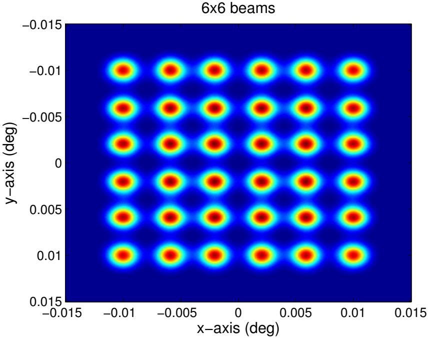
Multi-MaxSNR approach



- Sensitivity maximized in N_b (physical) directions
- Equivalent directivity pattern = discrete collection of sensitive beams
- Physical interpretation ⇒ single dish emulates N_b single dishes
- Statistically optimum for discrete collection of far-field point-sources (e.g. 5G)
- Possible application : targeted survey
- Sub-optimal for all other data model

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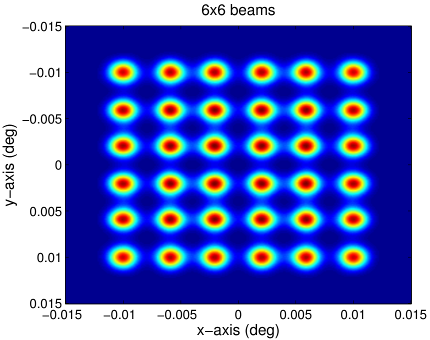

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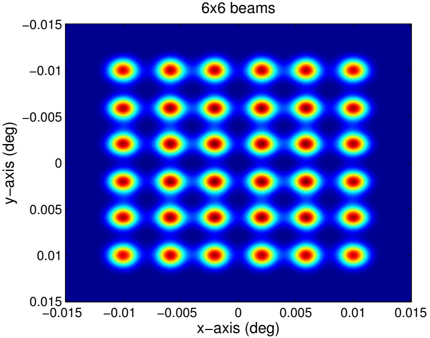

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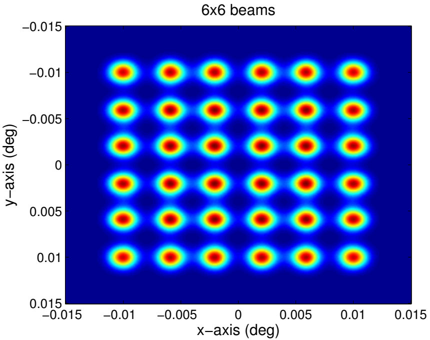

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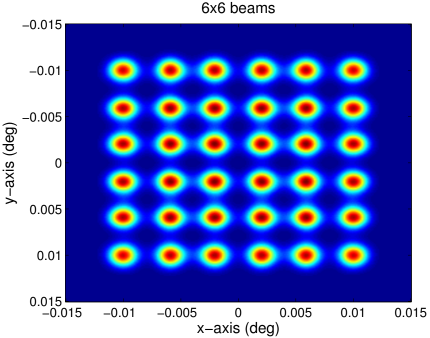

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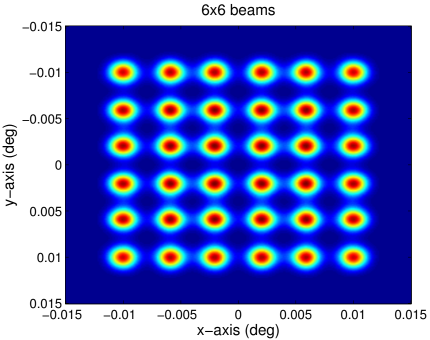

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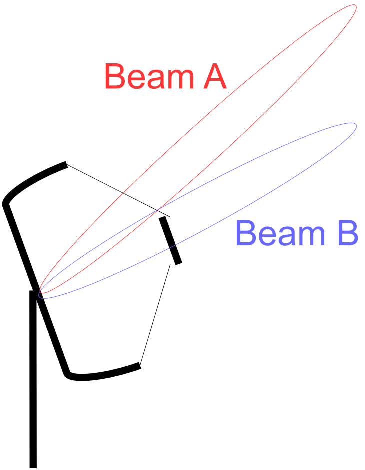
Information analysis



$$x_A(t) = \mathbf{w}_A^H \mathbf{x}(t)$$
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$$x_B(t) = \mathbf{w}_B^H \mathbf{x}(t)$$
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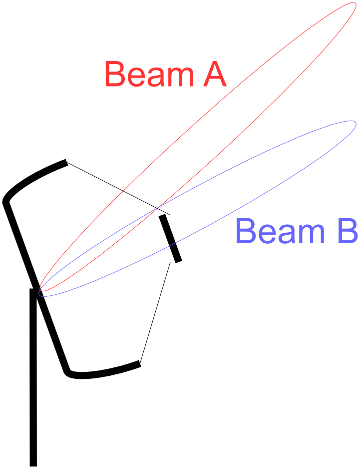
$$\mathbb{E}\left\{x_A(t)x_B^*(t)\right\} \neq 0$$

- Data spatial structure
- Beam-to-Beam correlation

Information analysis



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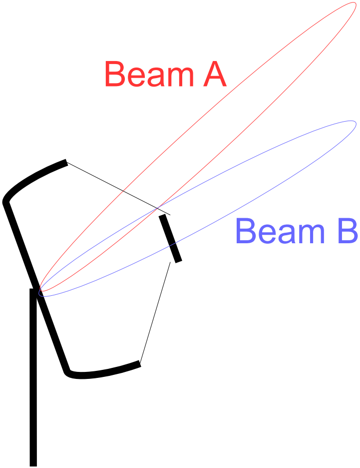

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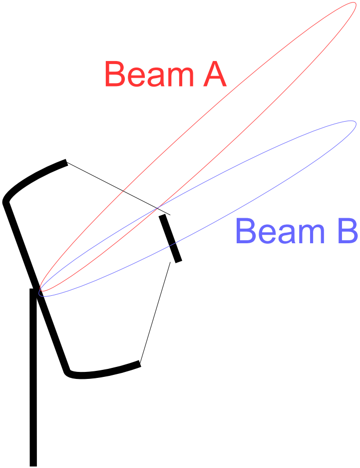

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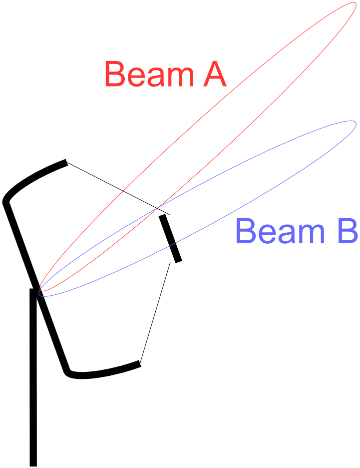

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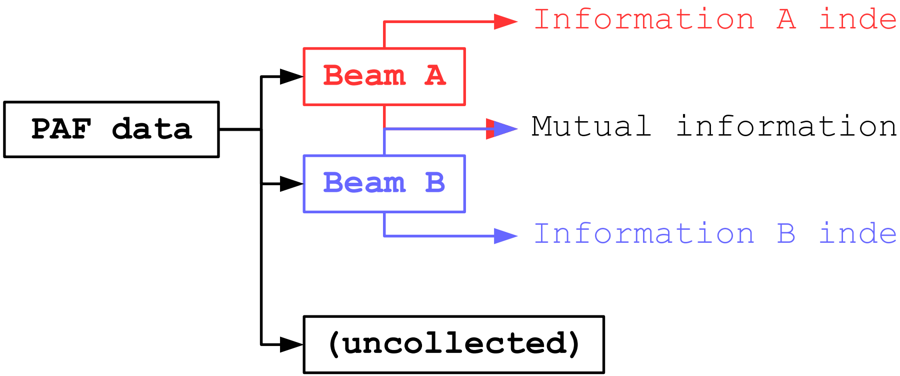

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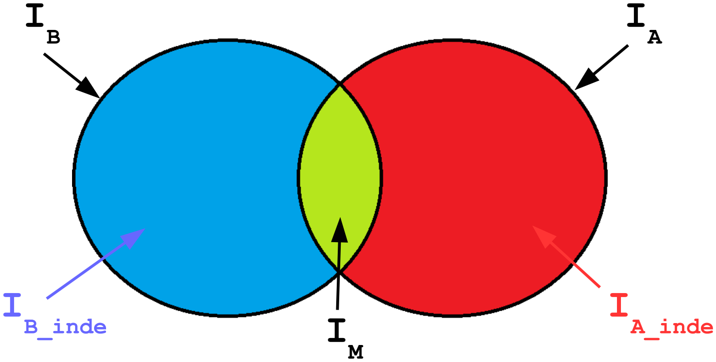

$$I_{\mathrm{TOT}} = I_{A_{\mathrm{inde}}} + I_{B_{\mathrm{inde}}} + 2I_M$$

$\Rightarrow$ I_M is collected twice

Information-wise efficiency

$$\begin{aligned} \text{PAF-instrument optimum} \quad &\Leftrightarrow \quad \text{no information redundancy} \\ &\Leftrightarrow \quad I_{M_{\mathrm{TOT}}} = 0 \\ &\Leftrightarrow \quad \text{independence between beams} \end{aligned}$$

Independent beam design induce Beam-to-Beam correlation:

spatially white field

$$R_{AB} = \mathbb{E}\left\{x_A(t)x_B^*(t)\right\} = \frac{\mathbf{w}_A{}^H \mathbf{w}_B}{\|\mathbf{w}_A\| \, \|\mathbf{w}_B\|}$$

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Geometrical interpretation

$$\frac{\mathbf{w}_A{}^H \mathbf{w}_B}{\|\mathbf{w}_A\| \, \|\mathbf{w}_B\|} = \frac{\|\mathbf{w}_A\| \, \|\mathbf{w}_B\| \cos\theta_{A,B}}{\|\mathbf{w}_A\| \, \|\mathbf{w}_B\|}$$

Complex angle:

$$\cos\theta_{A,B} = \rho e^{i\psi}$$

with:

- $\rho = \cos\theta_{H_{A,B}}$ and $\theta_{H_{A,B}}$ is the *Hermitian angle*
- ψ is *Kasner's pseudo-angle*

$$\rho = 0 \Rightarrow \text{orthogonality}$$
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$$\Leftrightarrow \cos\theta_{H_{A,B}} = 0$$
$$\Leftrightarrow \mathbf{w}_A \text{ and } \mathbf{w}_B \text{ are } \textbf{orthogonal}$$

Extend to multiple beams - define $\mathbf{R}_{\text{eff}}$:

$$\mathbf{R}_{\text{eff}} = \mathbf{W}^H\mathbf{W} = \begin{bmatrix} \|\mathbf{w}_1\|^2 & r^*_{1,2} & \cdots & r^*_{1,N_b} \\ r_{1,2} & \|\mathbf{w}_2\|^2 & & \\ \vdots & & \ddots & \\ r_{N_b,1} & & & \|\mathbf{w}_{N_b}\|^2 \end{bmatrix}$$

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Reminder : orthogonalization

$$\mathbf{R}_{\text{eff}} = \mathbf{W}^H\mathbf{W} = \mathbf{U}.\text{diag}\left\{\mathbf{s}\right\}.\mathbf{U}^H$$

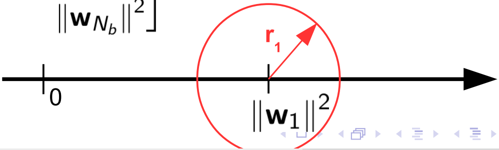
with:

$$\mathbf{s} = [\underbrace{\overbrace{s_1 \ , \ s_2 \ , \ \ldots \ , \ s_r}^{\text{rank}(\mathbf{R}_{\text{eff}})} \ , \ 0 \ , \ \ldots \ , \ 0}_{\dim(\mathbf{R}_{\text{eff}})}]^T$$

Gershgorin circles:

$\Sigma = r_1$

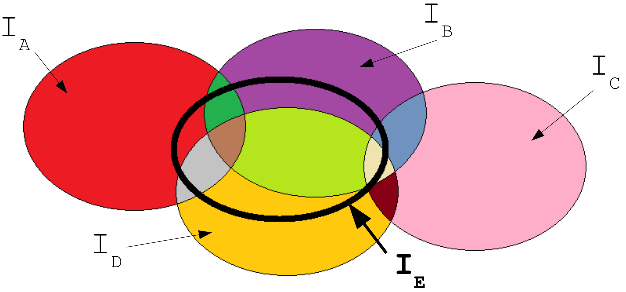
$$\begin{bmatrix} \left\|\mathbf{w}_1\right\|^2 & r_{1,2}^* & \cdots & r_{1,N_b}^* \\ r_{1,2} & \left\|\mathbf{w}_2\right\|^2 & & \\ \vdots & & \ddots & \\ r_{N_b,1} & & & \left\|\mathbf{w}_{N_b}\right\|^2 \end{bmatrix}$$



"Efficiency" analysis

Case 1: dim($\mathbf{R}_{eff}$) > rank($\mathbf{R}_{eff}$)

- 0 is eigenvalue with multiplicity dim($\mathbf{R}_{eff}$) - rank($\mathbf{R}_{eff}$)
- dim($\mathbf{R}_{eff}$) - rank($\mathbf{R}_{eff}$) vectors are linear combination of others
- Same "pattern" can be obtained with less beams (LCMV...)



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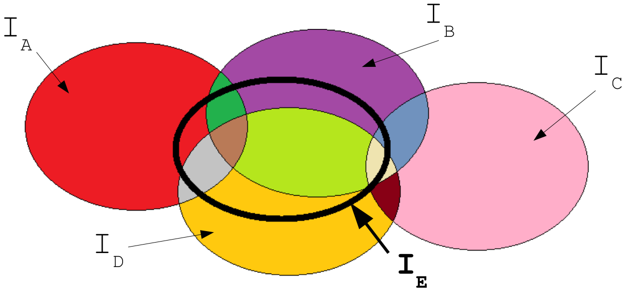

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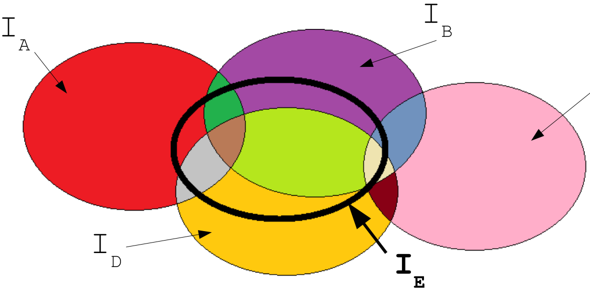

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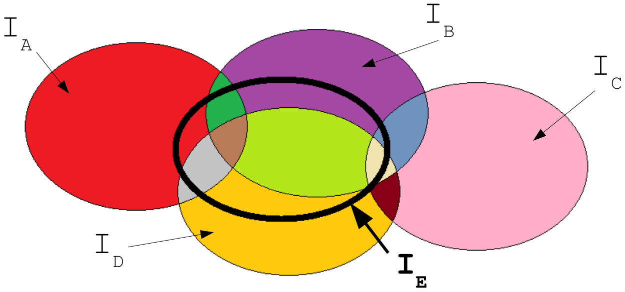

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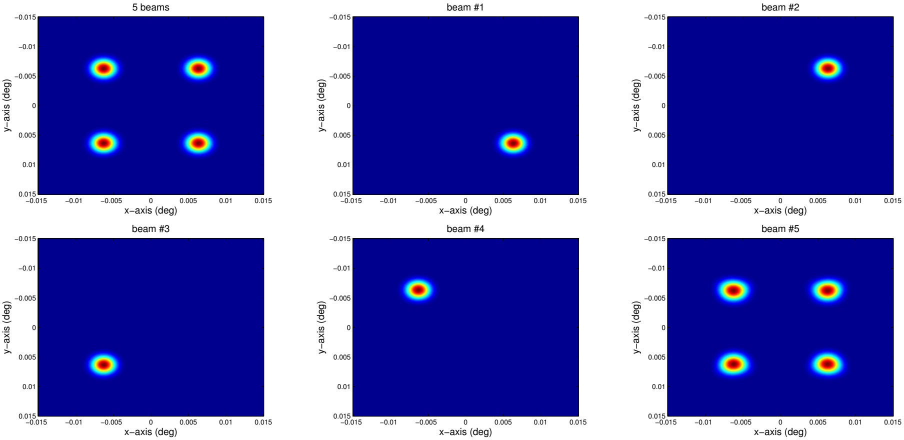


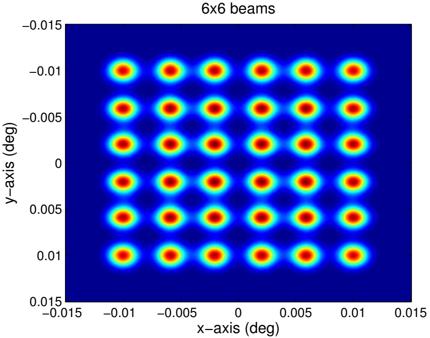
(information redundancy - unnecessary in radio astronomy, maybe RFI analysis)

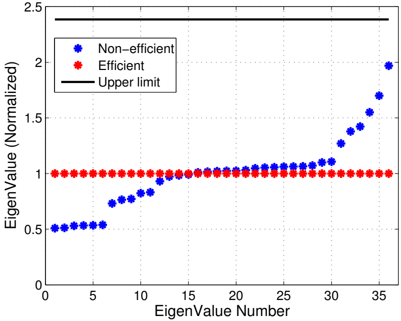
"Efficiency" analysis

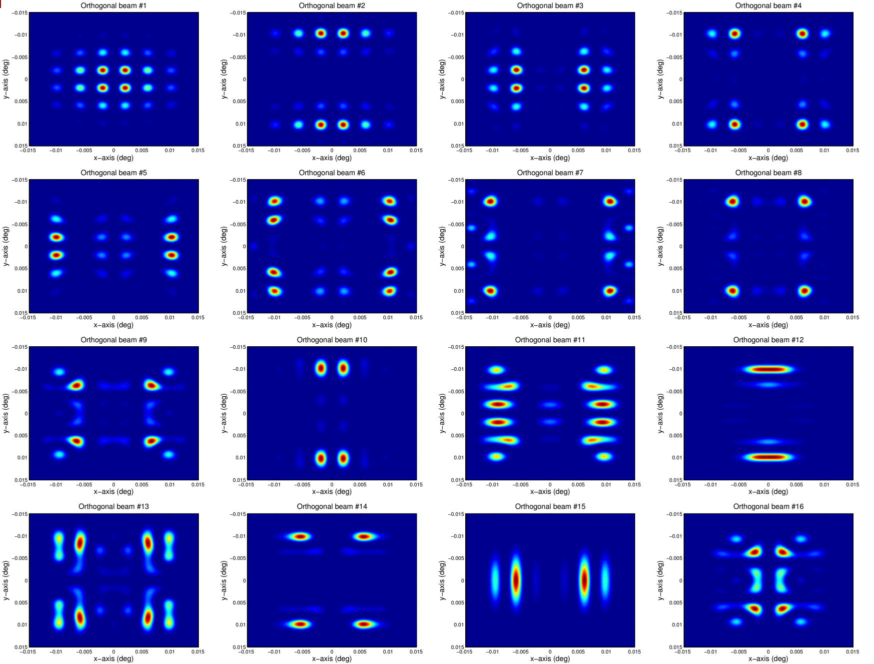
Case 2: $dim(\mathbf{R}_{eff}) = rank(\mathbf{R}_{eff})$

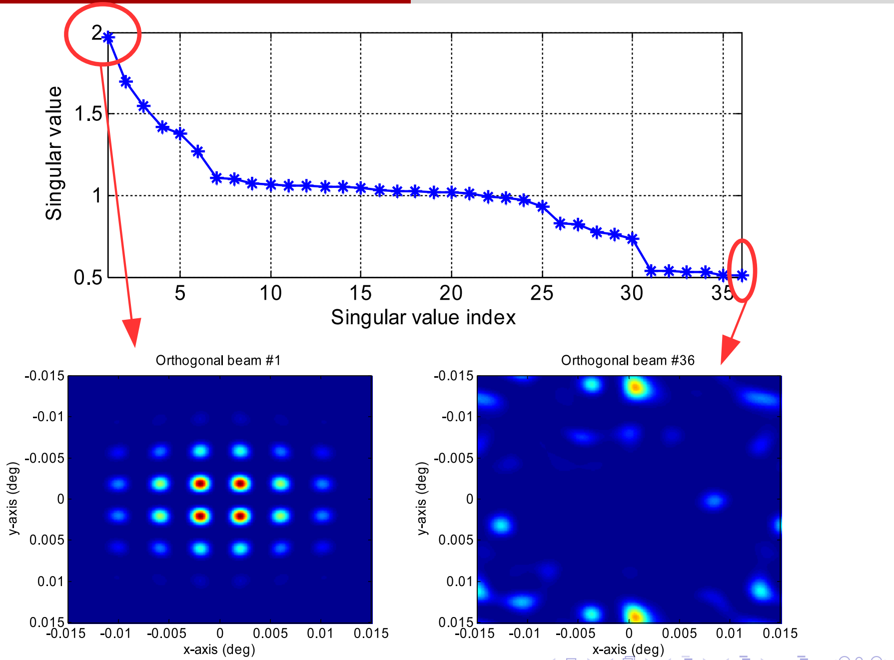
- maximum efficiency obtained if $\mathbf{R}_{\text{eff}}$ is diagonal
- need a metric to evaluate efficiency : $\kappa = \frac{\lambda_{\text{max}}}{\lambda_{\text{min}}}$

Example:



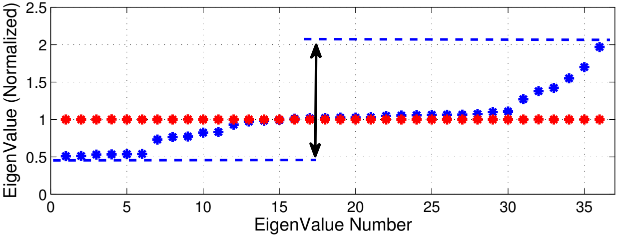



Orthogonal beam #1
Orthogonal beam #2
Orthogonal beam #3
Orthogonal beam #4
Orthogonal beam #5
Orthogonal beam #6
Orthogonal beam #7
Orthogonal beam #8
Orthogonal beam #9
Orthogonal beam #10
Orthogonal beam #11
Orthogonal beam #12
Orthogonal beam #13
Orthogonal beam #14
Orthogonal beam #15
Orthogonal beam #16
x−axis (deg)
y−axis (deg)



Singular value
Singular value index
Orthogonal beam #1
Orthogonal beam #36
y-axis (deg)
x-axis (deg)

Efficiency metric



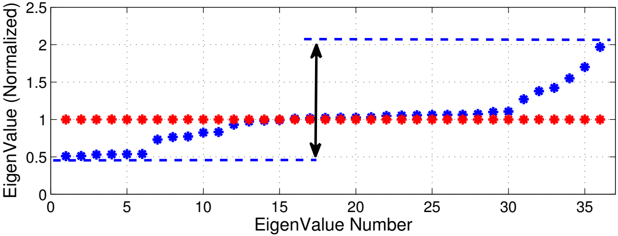
Condition number:

$$1 \leq \kappa \leq \frac{\text{Upper}\left\{\lambda_{\text{max}}\right\}}{\text{Lower}\left\{\lambda_{\text{min}}\right\}}$$

$$\text{Upper}\left\{\lambda_{\text{max}}\right\} = \underset{j}{max} \sum_i \left|\mathbf{R}_{\text{eff}_{i,j}}\right| \qquad \text{Lower}\left\{\lambda_{\text{min}}\right\} = \max\left\{\underset{j}{Max} \sum_i \left|\mathbf{R}_{\text{eff}_{i,j}}\right|, 0\right\}$$

$$I_M = 0 \Leftrightarrow \lambda_{\text{max}} = \lambda_{\text{min}} \Leftrightarrow \kappa = 1$$

Efficiency metric



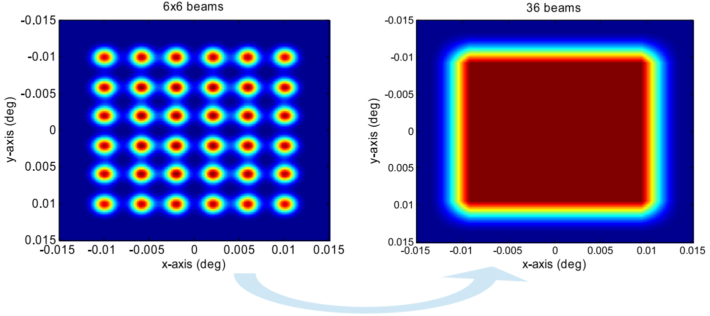
Condition number:

$$1 \leq \kappa \leq \frac{\text{Upper}\left\{\lambda_{\max}\right\}}{\text{Lower}\left\{\lambda_{\min}\right\}}$$

$$\text{Upper}\left\{\lambda_{\max}\right\} = \max_{j} \sum_i \left|\mathbf{R}_{\text{eff}_{i,j}}\right| \qquad \text{Lower}\left\{\lambda_{\min}\right\} = \max\left\{ \max_{j} \sum_i \left|\mathbf{R}_{\text{eff}_{i,j}}\right|, 0 \right\}$$

$$I_M = 0 \iff \lambda_{\max} = \lambda_{\min} \iff \kappa = 1$$

"Symbiotic" beamforming



$$\mathbf{W}_{\text{sym}} = \underset{\mathbf{W}}{\operatorname{argmax}} \int\int \mathbf{SNR}\left(\mathbf{W}, \theta, \phi\right) d\theta d\phi$$
$$\text{subject to} \quad \kappa = 1$$

Designing a Symbiotic beamformer

- ?
- work on mathematical formulation and optimization
- uniqueness of solution not ensured (not likely)
- requires further work for imaging (interferometer)
- should provide relationship between # of beams and Field-of-View

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